MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2018

Calculator-assumed

Marking Key

The release date for this exam and marking scheme is

- **the end of week 8 of term 2, 2018**

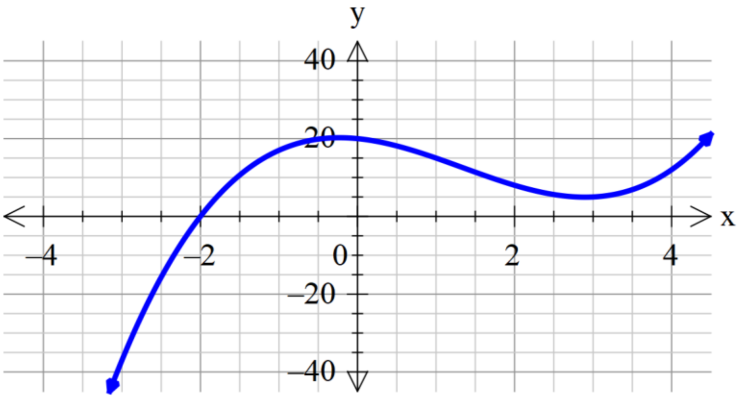
Question 8 **(5 marks)**

Solution
If $\lvert x \rvert = x^2 + x - 8$ then if $x > 0$ we have $x = x^2 + x - 8 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8}$. Since we have assumed that $x > 0$ we reject the negative square root. If $x < 0$ then $-x = x^2 + x - 8 \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow (x+4)(x-2) = 0 \Rightarrow x = 2, -4$ Since we have assumed that $x < 0$ we reject the positive root. Hence the solution is $x = \sqrt{8}$ or -4.

Mathematical behaviours	Marks
• calculates the correct root assuming that $x > 0$ (2 marks)	1
• calculates the correct root assuming that $x < 0$ (2 marks)	1
• states the overall solution correctly	1

Question 9 (a) **(3 marks)**

Solution



From the graph it appears that $p(x)$ has a zero at $x=-2$

We evaluate $p(-2)=(-2)^3-4(-2)^2-2(-2)+20=-8-16+4+20=0$

So by the factor theorem $z-(-2)=z+2$ is a factor of $p(z)$

Mathematical behaviours	Marks
• obtains $x=-2$ as a possible zero of *p*	1
• demonstrates that $p(-2)=0$	1
• deduces that $z+2$ is a factor of $p(z)$	1

Question 9(b) **(3 marks)**

Solution

Long division gives that $z^3-4z^2-2z+20=(z+2)(z^2-6z+10)$

If $z^2-6z+10=0 \Rightarrow z=3\pm i$ by either using the quadratic formula or completing the square

Hence $z-3-i$ and $z-3+i$ are the conjugate linear factors of $p(z)$

Mathematical behaviours	Marks
• identifies the correct quadratic factor	1
• determines correctly the zeros of the quadratic factor	1
• states correctly the corresponding conjugate linear factors	1

Question 10(a) **(2 marks)**

Solution	
$\mathbf{v}(0) = 200\cos\frac{\pi}{3}\mathbf{i} + 200\sin\frac{\pi}{3}\mathbf{j}$ ms^{-1} $= 100\mathbf{i} + 100\sqrt{3}\mathbf{j}$ ms^{-1}	
Mathematical behaviours	**Marks**
• correctly states the horizontal and vertical components of $\mathbf{v}(0)$ • simplifies correctly	1 1

Question 10(b) **(2 marks)**

Solution	
The acceleration is $\mathbf{a}(t) = -9.8\mathbf{j}$ ms^{-2}, $t > 0$ $\Rightarrow \mathbf{v}(t) = -9.8t\,\mathbf{j} + \mathbf{c}$ $\mathbf{v}(0) = 100\mathbf{i} + 100\sqrt{3}\mathbf{j}$ ms^{-1} $\Rightarrow \mathbf{c} = 100\mathbf{i} + 100\sqrt{3}\mathbf{j}$ $\therefore \mathbf{v}(t) = 100\mathbf{i} + (100\sqrt{3} - 9.8t)\mathbf{j}$ ms^{-1}	
Mathematical behaviours	**Marks**
• integrates $\mathbf{a}(t)$ to determine $\mathbf{v}(t)$ • determines the constant and states the correct $\mathbf{v}(t)$	1 1

Question 10(c) **(6 marks)**

Solution	
$\mathbf{r}(t) = \int \mathbf{v}(t)dt = 100t\,\mathbf{i} + (100\sqrt{3}t - 4.9t^2)\mathbf{j}$ m maximum height $\Rightarrow 100\sqrt{3} - 9.8t = 0 \Rightarrow t = 17.7$ seconds (rounded to 1 decimal place) $\therefore$ maximum height is $\Rightarrow 1530.6$ metres (rounded to 1 decimal place) Maximum horizontal distance when vertical distance is back to 0 i.e. when $100\sqrt{3}t - 4.9t^2 = 0 \Rightarrow t(100\sqrt{3} - 4.9t) = 0$ i.e. when $100\sqrt{3} - 4.9t = 0 \Rightarrow t = \frac{100\sqrt{3}}{4.9} = 35.35$ seconds $\therefore$ Maximum horizontal distance $= 100(35.35) = 3535$ m	
Mathematical behaviours	**Marks**
• integrates $\mathbf{v}(t)$ to determine $\mathbf{r}(t)$ (accept assumption of constants of integration as 0 from initial conditions, without statement) • uses vertical component of $\mathbf{v}(t) = 0$ determine the time when max height is reached • determines the max height • recognises that the max horizontal distance when vertical component =0 • determines the time when max distance is reached • determines max distance	1 1 1 1 1 1

Question 11 (a) **(2 marks)**

Solution	
Now $\lvert z_1 \rvert = \sqrt{9^2+2^2} = \sqrt{85}$ and $\lvert z_2 \rvert = \sqrt{(-7)^2+6^2} = \sqrt{85}$ Since a pair of adjacent sides in the parallelogram *P* have the same length, all sides have the same length. Hence P is a rhombus.	
Mathematical behaviours	Marks
• calculates the correct values of $\lvert z_1 \rvert$ and $\lvert z_2 \rvert$ • makes a valid deduction about the lengths of all sides of P	1 1

Question 11 (b) **(3 marks)**

Solution	
Denote $\arg z_1 = \varphi_1$. Then $\tan \varphi_1 = \frac{2}{9} \Rightarrow \varphi_1 \approx 0.2187$ Denote $\arg z_2 = \varphi_2$. Then $\tan \varphi_2 = \frac{6}{(-7)} \Rightarrow \varphi_2 \approx 2.4330$ Hence $\varphi_2 - \varphi_1 \approx 2.4330 - 0.2187 = 2.2143$ So the required angle is 2.21 radians (correct to 2 decimal places)	
Mathematical behaviours	Marks
• determines correctly the value of φ_1 • determines correctly the value φ_2 • gives the required answer to the prescribed accuracy	1 1 1

Question 11 (c) **(2 marks)**

Solution	
We see that $z_1 - z_2 = (9+2i)-(-7+6i) = 16-4i$ and that $-2i(z_1+z_2) = -2i(2+8i) = 16-4i$ Hence it follows that $z_1 - z_2 = -2i(z_1+z_2)$	
Mathematical behaviours	Marks
• evaluates correctly the value of $z_1 - z_2$ • evaluates correctly the value of $-2i(z_1+z_2)$.	1 1

Question 11 (d) **(4 marks)**

Solution
The diagonals of P are $z_1 + z_2$ and $z_1 - z_2$. From part (c) we deduce that $\lvert z_1 - z_2 \rvert = \lvert -2i \rvert \times \lvert z_1 + z_2 \rvert = 2 \lvert z_1 + z_2 \rvert$ …….(i) Hence one diagonal is twice as long as the other. Also we have that $\arg(z_1 - z_2) = \arg(-2i) + \arg(z_1 + z_2) = -\frac{\pi}{2} + \arg(z_1 + z_2)$ ………(ii) Hence the angle between the diagonals is $\frac{\pi}{2}$.

Mathematical behaviours	Marks
• derives the result (i)	1
• deduces the correct ratio of the lengths of the diagonals	1
• derives the result (ii)	1
• deduces the correct angle between the diagonals	1

Question 12(a) **(3 marks)**

Solution	
Since the function involves a square root we need to check that the quadratic is positive for all real values of x. As $$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$$ the function is defined for all real values	
Mathematical behaviours	**Marks**
• notes that we need to show that the contents of the square root is positive • either completes the square or shows the quadratic never equals zero • deduces the required result	1 1 1

Question 12(b) **(3 marks)**

Solution	
If $f(\alpha)=f(\beta)$ then $$\alpha^2+\alpha+1=\beta^2+\beta+1\Rightarrow(\alpha^2-\beta^2)+(\alpha-\beta)=0$$ This then implies that $(\alpha-\beta)(\alpha+\beta+1)=0\Rightarrow\alpha=\beta$ or $\alpha+\beta=-1$	
Mathematical behaviours	**Marks**
• forms an equation without the square root expressing the fact that $f(\alpha)=f(\beta)$ • solves the equation • states the two possible solutions	1 1 1

Question 12(c) **(3 marks)**

Solution	
The function is not one-to-one. If the function were 1-1 then $f(\alpha)=f(\beta)$ would force $\alpha=\beta$. Since here we have the possibility that $\alpha+\beta=-1$ we do not have a 1-1 function	
Mathematical behaviours	**Marks**
• states the correct conclusion • indicates the property that must be satisfied by a 1-1 function • justifies why this function does not possess this property	1 1 1

Question 13 **(6 marks)**

Solution
If $z^4 = -100$ then $\lvert z^4 \rvert = \lvert z \rvert^4 = 100 \Rightarrow \lvert z \rvert = 100^{1/4} = \sqrt{10}$. Also $\arg z^4 = 4\arg z = \arg(-100) = \pi \ (\text{mod}\, 2\pi)$ so that $4\arg z = (1+2k)\pi \Rightarrow \arg z = \dfrac{(2k+1)\pi}{4}$ for $k = 0,1,2,3$. Hence $\arg z = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$ or $\dfrac{7\pi}{4}$. Written in polar form, $z = \sqrt{10}$ cis ϑ with $\vartheta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$ or $\dfrac{7\pi}{4}$. Moreover, cis $\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}(1+i)$, cis $\dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}}(-1+i)$, cis $\dfrac{5\pi}{4} = \dfrac{1}{\sqrt{2}}(-1-i)$, cis $\dfrac{7\pi}{4} = \dfrac{1}{\sqrt{2}}(1-i)$, and $\dfrac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$ so the solutions are $z_1 = \sqrt{5}(1+i)$. $z_2 = \sqrt{5}(-1+i)$, $z_3 = \sqrt{5}(-1-i)$ and $z_4 = \sqrt{5}(1-i)$.

Mathematical behaviours	Marks
• derives the result that $\lvert z \rvert^4 = 100$	1
• deduces the correct value of $\lvert z \rvert$	1
• obtains one value of $\arg z = \pi / 4$	1
• determines all four possible values of $\arg z$	1
• writes down the the real and imaginary parts of the four ‘‘cis’ values	1
• solves correctly for the real and imaginary parts of all four solutions	1

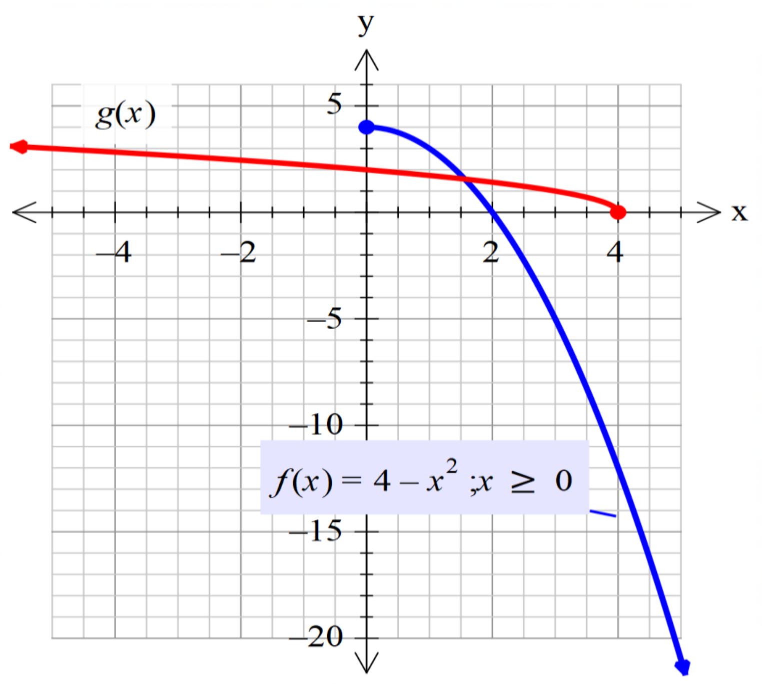
Question 14(a) **(3 marks)**

Solution	
If $y = 4 - x^2 \Rightarrow x^2 = 4 - y \Rightarrow x = \sqrt{4-y}$ Therefore the inverse function is $g(x) = \sqrt{4-x}$ (Alternatively students may define the other branch with $x = -\sqrt{4-y}$ so $g(x) = -\sqrt{4-x}$.)	
Mathematical behaviours	**Marks**
• writes the equation $x = f(y)$ • solves for x • states correct inverse function	1 1 1

Question 14(b) **(2 marks)**

Solution	
Domain is $x \in (-\infty, 4]$ Range is $[0, \infty)$ (If student took the alternative definition the range becomes $(-\infty, 0]$.)	
Mathematical behaviours	**Marks**
• states correct domain • states correct range	1 1

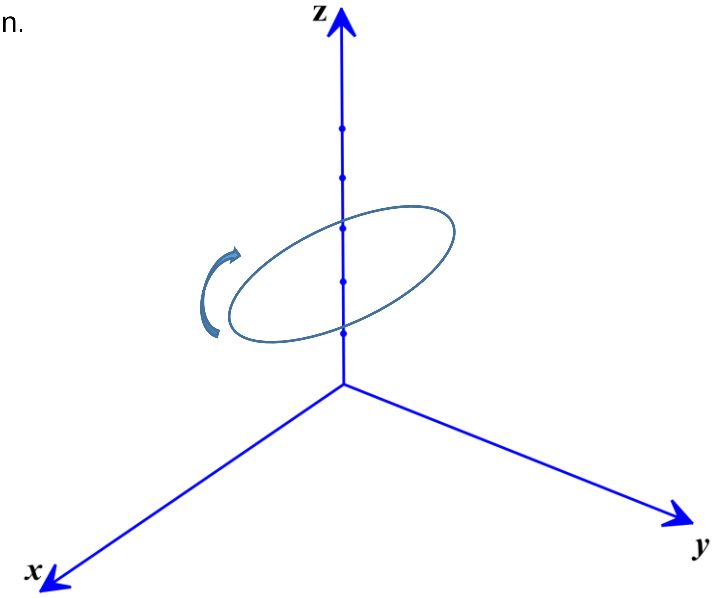
Question 14(c) **(2 marks)**

Solution	
	
Mathematical behaviours	**Marks**
• gives neat sketch of $f(x)$ • gives neat sketch of $g(x)$	1 1

Question 14(d) **(2 marks)**

Solution	
The two graphs are reflections of each other in the line $y = x$	
Mathematical behaviours	**Marks**
States the correct geometrical relationship with marks • for mentioning reflection • for giving the equation of the line of reflection	 1 1

Question 15(a)(b) **(3+2 marks)**

Solution
$x = \cos(2t)$ $y = \sin(2t)$ since $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$ $z = 2$ Point P moves in a horizontal circle, +2 units above the $xy-$plane as indicated by the $2\mathbf{k}$. The centre of the circle is (0,0,2) and the radius is 1 unit. At $t = 0$, $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{k}$ so the particle starts at (1,0,2) and at $t = \frac{\pi}{4}$, $\mathbf{r}(t) = \mathbf{j} + 2\mathbf{k}$ so the particle moves in an anticlockwise direction. 

Mathematical behaviours	Marks
• indicates that the particle moves in plane parallel to the $xy-$plane and 2 units above it	1
• indicates that it is a circle	1
• states the centre and radius of the circle	1
• indicates that the particle moves in an anticlockwise direction	1
• provides an appropriately labelled diagram of the circle in roughly the right position	1

Question 15(c) **(2 marks)**

Solution	
$d=\sqrt{\cos^2(2t)+\sin^2(2t)+2^2}$ $=\sqrt{1+4}$ $=\sqrt{5}$ The particle remains at a constant distance from the origin.	
Mathematical behaviours	Marks
• determines the correct distance • states that the distance from the origin is constant over time	1 1

Question 15(d) **(2 marks)**

Solution	
The position vector is given by $\boldsymbol{p}(\mathrm{t}) = \cos(2\mathrm{t})\,\boldsymbol{i} + \sin(2\mathrm{t})\boldsymbol{j} + 2\boldsymbol{k}$ Differentiating $\Rightarrow \mathbf{v}_p(t) = -2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j} + 0\mathbf{k}$	
Mathematical behaviours	Marks
• determines the correct velocity • (allow one mark if attempts to differentiate but makes an error)	2

Question 15(e) **(2 marks)**

Solution	
Given Q moves in a circle in the $xz-$plane , with centre (0,2) and radius 2, the vector equation is $\lvert x\mathbf{i}+(z-2)\mathbf{k}\rvert = 2 \Rightarrow x^2+(z-2)^2=4$ with $y=2$	
Mathematical behaviours	Marks
• states the correct cartesian equation • remembers to state that $y=2$	1 1

Question 15(f) **(3 marks)**

Solution	
$\left\lvert \mathbf{p}(\frac{\pi}{2})-\mathbf{q}(\frac{\pi}{2})\right\rvert = \left\lvert \cos(\pi)\mathbf{i}+\sin(\pi)\mathbf{j}+2\mathbf{k}-(2\cos(\frac{\pi}{2})\mathbf{i}+2\mathbf{j}+(2-2\sin(\frac{\pi}{2}))\mathbf{k}\right\rvert$ $=\lvert -\mathbf{i}+2\mathbf{k}-2\mathbf{j}\rvert$ $=\sqrt{(-1)^2+2^2+2^2}$ $=3$ units	
Mathematical behaviours	Marks
• indicates the need to determine $\left\lvert \mathbf{p}(\frac{\pi}{2})-\mathbf{q}(\frac{\pi}{2})\right\rvert$ • determines $\lvert -\mathbf{i}+2\mathbf{k}-2\mathbf{j}\rvert$ • calculates the correct distance	1 1 1

Question 16(a) **(4 marks)**

Solution
By De Moivre's theorem we have $$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4 .$$ Expanding and taking real parts gives $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ $= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$ $= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$ $= 8\cos^4\theta - 8\cos^2\theta + 1$

Mathematical behaviours	Marks
• uses De Moivre's theorem appropriately	1
• expands the fourth power of the expression correctly	1
• takes the real parts of each side	1
• replaces $\sin^2\theta = 1-\cos^2\theta$ and simplifies to obtain the result	1

Question 16 (b) **(5 marks)**

Solution
Let $x = \cos\theta$ where $0 \le \theta \le \pi$. Then by part (a) we have $p(x) = p(\cos\theta) = \cos 4\theta$ Hence the maximum and minimum values of $p(x)$ are ± 1 . At maximum values $\cos 4\theta = 1 \Rightarrow 4\theta = 2n\pi$ so that $\theta = 0, \frac{\pi}{2}$ or π within the range. Since $x = \cos\theta$ we have $x = 0, \pm 1$. At minimum values $\cos 4\theta = -1 \Rightarrow 4\theta = (2n+1)\pi$ so that $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ within range whence $x = \pm\frac{1}{\sqrt{2}}$. In summary, the maximum value of $p(x)$ is 1, and occurs at $x = 1, 0$ or -1 and the minimum value of $p(x)$ is -1, and occurs at $x = \pm 1/\sqrt{2}$

Mathematical behaviours	Marks
• derives 1 and -1 as the extreme values of $p(x)$	1
• determines the correct values of θ at the maximum	1
• infers the corresponding correct values of x	1
• determines the correct values of θ at the minimum	1
• infers the corresponding correct values of x	1

Question 17(a) **(4 marks)**

Solution	
We have $(f\circ g)(x)=3\left(x+\frac{1}{x}\right)-1=3x-1+\frac{3}{x}$. Domain is $x\neq 0$ and range is $\mathbb{R}$ (all real numbers) Similarly $(g\circ f)(x)=3x-1+\frac{1}{3x-1}$. Domain is $x\neq\frac{1}{3}$ and range is $\mathbb{R}$ (all real numbers)	
Mathematical behaviours	**Marks**
• determines the two composite functions correctly (one mark for each) • states domain and range of $(f\circ g)$ • states domain and range of $(g\circ f)$	1 1 1

Question 17(b) **(2 marks)**

Solution	
If $(f\circ g)(p)=(g\circ f)(p)$ then $3p-1+\frac{3}{p}=3p-1+\frac{1}{3p-1}\Rightarrow 3(3p-1)=p\Rightarrow p=\frac{3}{8}$	
Mathematical behaviours	**Marks**
• forms an appropriate equation for p • solves correctly	1 1

Question 18(a) **(1 mark)**

Solution	
Since plane $DEFG$ is parallel to the $yz-$plane, and $x=3$ everywhere on the plane the equation is $x=3$ (in vector form the equation is **r.i** $=3$**)**	
Mathematical behaviours	Marks
• states the correct equation of the plane	1

Question 18(b) **(3 marks)**

Solution	
Largest sphere will have a diameter = 3 units, so the radius = 1.5 units and centre = $\left(\frac{3}{2},2,\frac{c}{2}\right)$ Vector form: $\lvert\mathbf{r}-c\rvert=\rho \Rightarrow \left\lvert\mathbf{r}-\left(\frac{3}{2},2,\frac{c}{2}\right)\right\rvert=\frac{3}{2}$ Cartesian form: $\left(x-\frac{3}{2}\right)^2+(y-2)^2+\left(z-\frac{c}{2}\right)^2=\frac{9}{4}$	
Mathematical behaviours	Marks
• states the centre and radius of the sphere • states the vector equation • states the Cartesian equation	1 1 1

Question 18(c) **(3 marks)**

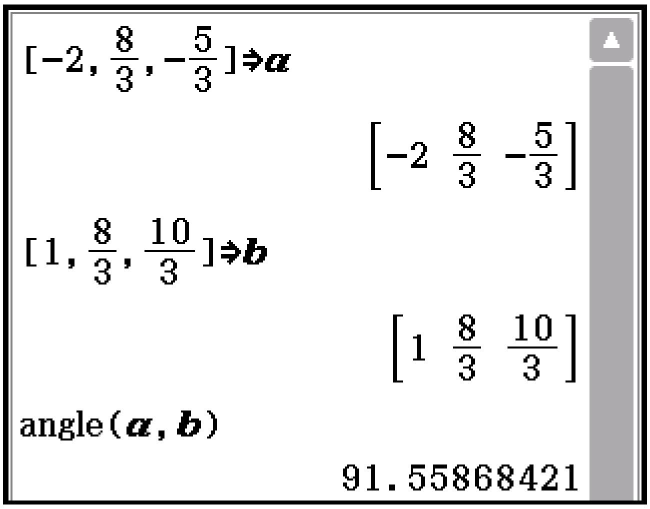
Solution	
$\overrightarrow{OB}=\begin{pmatrix}0\\4\\c\end{pmatrix}$ and $\overrightarrow{OG}=\begin{pmatrix}3\\4\\c\end{pmatrix} \Rightarrow \overrightarrow{OB}\times\overrightarrow{OG}=\begin{pmatrix}0\\3c\\-12\end{pmatrix}$ $\therefore$ equation of plane is: $0.(x-0)+3c.(y-0)-12.(z-0)=0$ $\Rightarrow \quad 3cy-12z=0$ $\Rightarrow \quad cy=4z$	
Mathematical behaviours	Marks
• determines the cross product of two appropriate vectors • uses the cross product to determine the equation of the plane • states the correct plane equation	1 1 1

Question 18(d) **(5 marks)**

Solution
(i) M has coordinates $\left(3,0,\frac{c}{2}\right)$ and $C(0,4,0)$ vector equation of a line L passing through points with position vectors $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{r}(t)=\mathbf{a}+t(\mathbf{b}-\mathbf{a})=<0,4,0>+t\left(<3,0,\frac{c}{2}>-<0,4,0)\right)$ $=<0,4,0>+t<3,-4,\frac{c}{2}>$ which in parametric form is: $x=3t,\ y=4(1-t),\ z=\frac{ct}{2}$
(ii) Substituting $x=3t,\ y=4(1-t),\ z=\frac{ct}{2}$ into $cy=4z$ $\Rightarrow 4c(1-t)=4\frac{ct}{2}$ $\Rightarrow \quad 4-4t=2t$ $\Rightarrow \quad t=\frac{2}{3}$ when $t=\frac{2}{3}, \quad x=2, y=\frac{4}{3}, z=\frac{c}{3}$ $\therefore$ the line intersects the plane at $\left(2,\frac{4}{3},\frac{c}{3}\right)$ so its position vector is $2\,\mathbf{i}+\frac{4}{3}\,\mathbf{j}+\frac{c}{3}\,\mathbf{k}$

Mathematical behaviours	Marks
• determines the position vector for M	1
• uses the position vectors of M and C to determine the vector equation of the line	1
• states the equation of the line in equivalent parametric form.	1
• substitutes into the equation of the plane $cy=4z$ (or whatever found in part (c))	1
• states the position vector of the point of intersection	1

Question 18(e) **(3 marks)**

Solution
For $c=5$, the point of intersect from part (d) X is at $\left(2,\frac{4}{3},\frac{5}{3}\right)$ and $C(0,4,0)$ $\therefore \overrightarrow{XC} = -2\mathbf{i}+\frac{8}{3}\mathbf{j}-\frac{5}{3}\mathbf{k}$ and $\overrightarrow{XG} = \mathbf{i}+\frac{8}{3}\mathbf{j}+\frac{10}{3}\mathbf{k}$ Using the dot product of $\overrightarrow{XC}$ and $\overrightarrow{XG}$ or the angle between vectors on a CAS calculator the angle between the line and the plane is 91.6°.   

Mathematical behaviours	Marks
• determines an appropriate vector in the line (e.g $\overrightarrow{XC}$)	1
• determines an appropriate vector in the plane (e.g $\overrightarrow{XG}$)	1
• states the angle between the plane and the line (accept any suitable rounding)	1